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# ASYMPTOTIC SOLUTION OF THE AXISYMMETRIC CONTACT PROBLEM FOR AN ELASTIC LAYER OF INCOMPRESSIBLE MATERIAL†

V. M. ALEKSANDROV

Moscow

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The solution of the axisymmetric contact problem for an elastic layer made of incompressible material and clamped along the base is constructed by regular and singular asymptotic methods. © 2003 Elsevier Ltd. All rights reserved.

A similar problem for a layer of incompressible material was considered previously in [1, 2] using the same methods.

## 1. FORMULATION OF THE PROBLEM

Suppose an elastic layer occupies a region $0 \leqslant r < \infty$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant h$ (where $h$ is the thickness of the layer) in a cylindrical system of coordinates $r, \varphi, z$. The layer is made of an incompressible material (Poisson's ratio $\nu = 1/2$) and is rigidly clamped along the base. As is well known [1, 2], the axisymmetric contact problem for such a layer reduces to determining the contact pressure $q(r)$ from the following integral equation

$$\int_0^a q(\rho)K\left(\frac{\rho}{h}, \frac{r}{h}\right)\rho d\rho = 2Gh\delta(r), \quad 0 \le r \le a \tag{1.1}$$

$$K(\sigma, \tau) = \int_0^\infty L(u)J_0(\sigma u)J_0(\tau u)du, \qquad L(u) = \frac{\operatorname{sh}2u - 2u}{\operatorname{ch}2u + 1 + 2u^2} \tag{1.2}$$

Here $a$ is the radius of the contact area, $G$ is the shear modulus, $\delta(r)$ is the settlement of the layer surface in the contact area, and $J_0(x)$ is the Bessel function. The function $L(u)$ behaves at zero ($u \to 0$) and at infinity ($u \to \infty$) as follows:

$$L(u) = \frac{2}{3}u^3 - \frac{6}{5}u^5 + O(u^7), \quad L(u) = 1 + O(e^{-2u}) \tag{1.3}$$

It can be shown [1, 2] on the basis of these properties of the function $L(u)$ that the kernel (1.2) of integral equation (1.1) can be represented in the form

$$K(\sigma, \tau) = K_0(\sigma, \tau) - F(\sigma, \tau)$$
$$K_0(\sigma, \tau) = \frac{2}{\pi(\sigma + \tau)}\mathbf{K}(k), \quad k = \frac{2\sqrt{\sigma\tau}}{\sigma + \tau} \tag{1.4}$$

where $\mathbf{K}(k)$ is the complete elliptical integral of the first kind. The function $K_0(\sigma, \tau)$ contains a singularity of the form $-\ln|\sigma - \tau|$. The function $F(\sigma, \tau)$ is continuous with all derivatives with respect to the set of variables $\sigma, \tau$ in the quarter-plane $0 \leqslant \sigma, \tau < \infty$. In the square $0 \leqslant \sigma, \tau < 1$ it can be represented by the following series, which is absolutely and uniformly convergent with respect to set of variables

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$$F(\sigma, \tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}\sigma^{2i}\tau^{2j}; \quad b_{ij} = \frac{(m!)^2}{(i!)^2(j!)^2}a_m, \quad m = i+j \tag{1.5}$$

The constants $a_m$ are defined by the integrals

$$a_m = \frac{(-1)^m}{[(2m)!!]^2}\int_0^{\infty}[1-L(u)]u^{2m}du \tag{1.6}$$

Calculations give $a_0 = 1.770217$ and $a_1 = -0.957769$.

On the basis of representation (1.4) of the kernel $K(\sigma, \tau)$ it can be shown [1, 2] that the general solution of integral equation (1.1) has the following structure

$$q(r) = \omega(r)/\sqrt{a^2 - r^2} \tag{1.7}$$

where the function $\omega(r)$ satisfies the Hölder condition in the circle $r \leqslant a$ with index $0 < \alpha \leqslant 1$, if the function $\delta(r)$ is such that its derivative satisfies the Hölder condition in the circle $r \leqslant a$ with index $\beta$ where $\alpha \leqslant \beta \leqslant 1$.

## 2. THE SOLUTION FOR A RELATIVELY THICK LAYER

We will introduce a dimensionless geometrical parameter $\lambda = h/a$ and we will assume that $\lambda \geqslant 2$ (a relatively thick strip).

The general solution of Eq. (1.1) for this case is given by the formulae [1, 2]

$$\begin{aligned}
q(r) &= q_*(r) + \frac{2}{\pi a\sqrt{a^2 - r^2}}\times \\
&\times\int_0^a p(\xi)\left[A_0\left(1 + A_0 + A_0^2 + A_0^3 + \frac{2}{3}A_1\right) + A_1(1 + A_0)\left(\frac{2r^2}{a^2} + \frac{\xi^2}{a^2} - 1\right)\right]d\xi + O\left(\frac{1}{\lambda^5}\right) \\
P &= 2\pi\int_0^a q(\rho)\rho d\rho = \\
&= 4\int_0^a p(\xi)\left[1 + A_0 + A_0^2 + A_0^3 + A_0^4 + \frac{2}{3}A_0A_1 + A_1(1 + A_0)\left(\frac{\xi^2}{a^2} + \frac{1}{3}\right)\right]d\xi + O\left(\frac{1}{\lambda^5}\right) \\
q_*(r) &= \frac{2}{\pi}\left[\frac{p(a)}{\sqrt{a^2 - r^2}} - \int_r^a \frac{p'(\xi)d\xi}{\sqrt{\xi^2 - r^2}}\right], \quad p(x) = 2G\left[\delta(0) + x\int_0^x \frac{\delta'(\rho)d\rho}{\sqrt{x^2 - \rho^2}}\right] \\
A_0 &= \frac{2a_0}{\pi\lambda}, \quad A_1 = \frac{4a_1}{\pi\lambda^3}
\end{aligned} \tag{2.1}$$

where $P$ is the indenting force.

## 3. THE DEGENERATE SOLUTION FOR A RELATIVELY THIN LAYER

We will now assume that $\lambda \leqslant 2$ (a relatively thin strip). We will construct the degenerate (penetrating) solution of the problem for small values of $\lambda$.

We apply to both sides of integral equation (1.1) an operator with respect to $r$ of the form

$$\int_0^r \frac{d\eta}{\eta}\int_0^{\eta}(\ldots)\xi d\xi \tag{3.1}$$

As a result we will have

$$\int_0^a q(\rho) M\left(\frac{\rho}{h}, \frac{r}{h}\right) \rho d\rho = -\frac{2G}{h} g(r) \tag{3.2}$$

$$M(\sigma, \tau) = \int_0^\infty \frac{L(u)}{u^2} J_0(\sigma u) J_0(\tau u) du \tag{3.3}$$

$$g(r) = \int_0^r \delta(\rho) \ln\frac{r}{\rho} \rho d\rho + C_0 \tag{3.4}$$

In expression (3.4) for the function $g(r)$ we have dropped the arbitrary irregular term of the form $C_1 \ln r$.

Note that since the kernel (1.2) of Eq. (1.1) contained the singularity $-\ln|\sigma - \tau|$, the kernel (3.3) of Eq. (3.2) contains the singularity $(\sigma - \tau)^2 \ln|\sigma - \tau|$. As a consequence of this, the general solution of the integral equation has the structure [3, 4]

$$q(r) = \Omega(r)/(a^2 - r^2)^{3/2} \tag{3.5}$$

where the function $\Omega(r)$ is such that its derivative satisfies the Hölder condition in the circle $r \leqslant a$ with index $\alpha$.

As $\lambda \to 0$ the kernel of (3.3), by virtue of the first relation of (1.4), takes the form

$$M(\sigma, \tau) = \frac{2}{3} h^2 \delta(\rho, r) \tag{3.6}$$

where $\delta(\rho, r)$ is the axisymmetric delta function, defined by the integral [1, 2]

$$\delta(\rho, r) = \int_0^\infty \alpha J_0(\rho\alpha) J_0(r\alpha) d\alpha, \quad \alpha = \frac{u}{h} \tag{3.7}$$

Taking Eq. (3.6) into account we obtain that the degenerate solution of the problem for small values of $\lambda$ has the form

$$q(r) = -\frac{3G}{h^3} g(r) \tag{3.8}$$

The constant $C_0$ in expression (3.4) of the function $g(r)$ must be later obtained from the condition that the solution in the form (3.5) has the structure (1.7), i.e. from the condition

$$\Omega(a) = 0 \tag{3.9}$$

## 4. THE PRINCIPAL TERM OF THE ASYMPTOTIC FORM OF THE SOLUTION FOR A RELATIVELY THIN LAYER

We will construct a solution of the boundary-layer type for small values of $\lambda$ in the neighbourhood of the contour $r = a$ of the contact area.

It was shown in [1, 2] that a plane boundary layer in the neighbourhood of the contour of the contact area can be obtained from the Wiener–Hopf integral equation

$$\int_0^\infty \varphi(\tau) M_*(\tau - t) d\tau = -\pi f(t), \quad M_*(y) = \int_0^\infty \frac{L(u)}{u^3} \cos uy du$$
$$t = \frac{a - r}{h}, \quad \tau = \frac{a - \rho}{h}, \quad \varphi(t) = \frac{q(\rho)}{2G}, \quad f(t) = \frac{g(r)}{h^3} \tag{4.1}$$

where the function $M_*(y)$ behaves as $y^2 \ln|y|$ as $y \to 0$. The method of solving equations of the form (4.1) is well known [5].

It can be shown that the function $\varphi(t)$ will have the form

$$\varphi(t) = \Omega_*(t) t^{-3/2} \tag{4.2}$$

where the function $\Omega_*(t)$ is such that its derivative satisfies the Hölder condition when $0 \leqslant t \leqslant R < \infty$ with index $\alpha$.

To determine the constant $C_0$ in representation (3.4) we will have the condition

$$\Omega_*(0) = 0 \tag{4.3}$$

## 5. EXAMPLE

We will consider the case of a punch with a flat base $\delta(r) \equiv \delta = \text{const}$ impressed into a layer.

From the last formula of (2.1) we obtain that $p(x) = 2G\delta$, and the first three formulae of (2.1) are simplified considerably. We have from formula (3.4) $g(r) = \delta r^2/4 + C_0$, and the degenerate solution (3.8) has the form

$$q(r) = -\frac{3G}{h^3}\left(\frac{\delta r^2}{4} + C_0\right) \tag{5.1}$$

In order to construct an analytical expression for the function $\varphi(t)$ we will approximate the function $L(u)$, in accordance with the asymptotic formulae (1.4), by the expression

$$L_*(u) = \frac{u^3}{(u^2 + A^2)\sqrt{u^2 + B^2}}$$

where $A = 0.761310$ and $B = 2.588024$. The error of this approximation does not exceed 20% for all $0 \leqslant u < \infty$.

Condition (4.3) leads to the relation

$$C_0 = -\frac{1}{4}\delta a^2 (1 + D_1\lambda + D_2\lambda^2); \quad D_1 = \frac{2B + A}{AB}, \quad D_2 = \frac{4B - A}{4AB^2}$$

A boundary-layer type solution is given by the expression

$$\begin{aligned}
q(r) &= \frac{2G\delta}{h^3}\left[\frac{a^2}{4}(D_1\lambda + D_2\lambda^2)\varphi_0(t) + \frac{ha}{2}\varphi_1(t) - \frac{h^2}{4}\varphi_2(t)\right] \\
\varphi_0(t) &= \frac{3}{2}\operatorname{erf}(\sqrt{Bt}) + \frac{3}{2}\frac{e^{-Bt}}{\sqrt{\pi Bt}} \\
\varphi_1(t) &= \frac{3}{2}t\operatorname{erf}(\sqrt{Bt}) + \frac{e^{-Bt}}{\sqrt{\pi Bt}}\left(\frac{3}{2}t - \frac{3}{4B}\right) \\
\varphi_2(t) &= \operatorname{erf}(\sqrt{Bt})\left(\frac{3}{2}t^2 - \frac{27}{5}\right) + \frac{e^{-Bt}}{\sqrt{\pi Bt}}\left(\frac{3}{2}t^2 - \frac{A^2}{2}t - \frac{8B^2 + A^2}{4B}\right)
\end{aligned} \tag{5.2}$$

where $\operatorname{erf}(x)$ are probability integrals.

It can be shown that the boundary-layer type solution (5.2) is automatically matched to the degenerate solution (5.1) as one moves away from the contour $r = a$ into the depth of the contact area.

Using the degenerate solution, we obtain the relation between the indenting force $P$, defined by formula (2.1), and the indentation of the punch $\delta$

$$P = \frac{3\pi G\delta a^4}{4h^3}\left(\frac{1}{2} + D_1\lambda + D_2\lambda^2\right)$$

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## REFERENCES

1. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
2. ALEKSANDROV, V. M. and POZHARSKII, D. A., *Non-classical Three-dimensional Problems of the Mechanics of Contact Interactions of Elastic Bodies*. Faktorial, Moscow, 1998.
3. POPOV, G. Ya., *The Concentration of Elastic Stresses Near Punches, Cuts, Thin Inclusions and Supports*. Nauka, Moscow, 1982.
4. ZELENTSOV, V. B., The solution of some integral equations of mixed problems of the theory of the bending of plates. *Prikl. Mat. Mekh.*, 1984, **48**, 6, 983–991.
5. NOBLE, B., *Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations*. Pergamon Press, London, 1958.

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